

A COANALYTIC RANK ON SUPER-ERGODIC OPERATORS

MOHAMMED YAHDI

ABSTRACT. Techniques from Descriptive Set Theory are applied in order to study the Topological Complexity of families of operators naturally connected to ergodic operators in infinite dimensional Banach Spaces. The families of ergodic, uniform-ergodic, Cesaro-bounded and power-bounded operators are shown to be Borel sets, while the family of super-ergodic operators is shown to be either coanalytic or Borel according to specific structures of the Space. Moreover, trees and coanalytic ranks are introduced to characterize super-ergodic operators as well as spaces where the above classes of operators do not coincide.

1. INTRODUCTION

Let T be a bounded operator on an infinite dimensional Banach space X , and let $A_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k$ be the n^{th} -Cesaro-mean of T . Consider the following definitions:

- T is ergodic if the sequence $\{A_n\}_{n \geq 1}$ converges in the space of operators $L(X)$ equipped with the strong operator topology S_{op}
- T is uniformly ergodic if the sequence $\{A_n\}_{n \geq 1}$ converges in $L(X)$ equipped with its natural norm.
- T is weakly ergodic if for any $x \in X$, the sequence $\{A_n(x)\}_{n \geq 1}$ weakly converges in X .
- T is Cesaro-bounded if the norms of $\{A_n\}_n$ are uniformly bounded.
- T is power-bounded if the norms of $\{T^n\}_n$ are uniformly bounded.

The definition of a super-ergodic operator is introduced in [11] as the super property associated with ergodic operators. Let $\ell^\infty(X)$ be the Banach space of bounded sequences in X and let $\mathcal{C}_\mathcal{U}(X)$ be the subspace of sequences $\{x_n\}_n$ such that: $\lim_\mathcal{U} \|x_n\| = 0$. T is super-ergodic if for any ultrafilter \mathcal{U} on \mathbb{N} , the ultrapower operator $T_\mathcal{U}$ is ergodic on the

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ultraproduct $X_{\mathcal{U}} := \ell^\infty(X)/\mathcal{C}_{\mathcal{U}}(X)$, where

$$T_{\mathcal{U}}(\bar{x}) := \{Tx_n\}_n + \mathcal{C}_{\mathcal{U}}(X) \quad \text{for any } \bar{x} := \{x_n\}_n + \mathcal{C}_{\mathcal{U}}(X).$$

In [11] it was shown that uniformly ergodic implies super-ergodic, which in turn implies ergodic. Examples were given to show that these implications are strict. An interesting question is to determine the structures of the Banach space that place strong limits on these relationships that might be possible instead of just particular examples. Translations of certain mathematical concepts to families of sets, then examining their positions in the descriptive set hierarchy have been proven to be a productive approach (see [9] and [10]). For example, [10] investigates the “family of superstable operators” where its topological complexity is shown to be connected to the structure of the space; such as having some kind of unconditional basis or being hereditarily indecomposable. In particular, this produced families of spaces where stable, superstable and uniformly-stable operators are either equivalent or strictly separated. The important results in [8], [4] and [7] show the power of some applications of descriptive set theory.

In this work, some techniques from [10] are applied to sets related to classes of ergodic operators. For the particular class of super-ergodic operators, a characterization in terms of “trees” is introduced and more descriptive set theory tools are applied. It is shown that the height of these trees is a coanalytic rank, thus proving that the set of super-ergodic operators is coanalytic for the strong operator topology. All the other families of operators are shown to be Borel. In particular, this gives a general class of Banach spaces for which super-ergodic is strictly stronger than ergodic and strictly weaker than uniformly ergodic. Moreover, examples are given to show the existence of such spaces.

Throughout this work, X denotes an infinite dimensional Banach space with a norm $\|\cdot\|$.

2. APPLICATION OF DESCRIPTIVE SET THEORY

In a Polish space P , we consider the natural hierarchy Π_ξ^0 and Σ_ξ^0 covering all Borel subsets starting from the open and closed sets for $\xi = 1$ to more complex ones defined by induction on the countable ordinals ξ ; where a Σ_ξ^0 is a countable union of $\Pi_{\xi-1}^0$ sets, a Π_ξ^0 is a countable intersection of $\Sigma_{\xi-1}^0$ sets, and for α a limit ordinal:

$$\Sigma_\alpha^0 = \bigcup_{\xi < \alpha} \Pi_\xi^0 \quad \text{and} \quad \Pi_\alpha^0 = \bigcap_{\xi < \alpha} \Sigma_\xi^0$$

Borel sets are not the only subsets in a Polish space that are constructible from open and closed sets (see [2], [3] or [5]):

- $A \subset P$ is analytic if it is a continuous image of a Polish space.
- $A \subset P$ is coanalytic if $P \setminus A$ is analytic.

The space $L(X)$ of bounded operators equipped with the strong operator topology S_{op} is not a Polish space since it is not a Baire space. However, it is also possible to work in a standard Borel space; i.e., a space Borel isomorphic to a Borel set of a Polish space.

Proposition 2.1. *For any separable Banach space X , $L(X)$ of bounded operators is a standard Borel space when equipped with the σ -algebra $\sigma(S_{op})$ generated by the strong operator topology.*

Proof. Let X be a separable Banach space. It is well-known that with the strong operator topology the spaces $L_n(X)$ of operators of norm $\leq n$ is a Polish space in which $T \mapsto T^n$ is clearly continuous (see, e.g., page 14 in [5] or lemma 3. in [10]). So $L(X) = \bigcup_n L_{n+1}(X) \setminus L_n(X)$ is clearly standard Borel when equipped with the σ -algebra generated by the strong operator topology. \square

The following lemma is a modified result from [1].

Lemma 2.2. *Let T be a Cesaro-bounded operator on a Banach space X , and let $\{A_n\}_{n \in \mathbb{N}}$ be the corresponding sequence of Cesaro-means. Then T is ergodic if and only if the sequence $\{A_n x\}_n$ converges in norm for all x in a dense (in norm) subset of X .*

Proof. Put $E := \{x \in X : \{A_n x\}_{n \in \mathbb{N}} \text{ norm-converges in } X\}$. Suppose that E is dense in X and that $M := \sup_{n \in \mathbb{N}} \|A_n\| < \infty$.

Let $y \in X$ be fixed. For $\varepsilon > 0$, consider $x \in X$ such that $\|x - y\| < \varepsilon$. Then, there exists $n_0 \in \mathbb{N}$ such that $\|A_n x - A_m x\| < \varepsilon$, $\forall n, m \geq n_0$. Thus, for all $n, m \geq n_0$ we have

$$\begin{aligned} \|A_n y - A_m y\| &\leq \|A_n(y - x)\| + \|A_n x - A_m x\| + \|A_m(x - y)\| \\ &\leq (2M + 1)\varepsilon. \end{aligned}$$

Therefore the sequence $\{A_n y\}_{n \in \mathbb{N}}$ is Cauchy, and hence $y \in E$. \square

Denote by $\mathcal{E}(X)$, $\mathcal{SE}(X)$, $\mathcal{UE}(X)$, $L_{cb}(X)$ and $L_{pb}(X)$, respectively the subsets of $L(X)$ of ergodic, super-ergodic, uniformly ergodic, Cesaro-bounded and power bounded operators on X .

Proposition 2.3. *For any separable Banach space X , the sets $L_{pb}(X)$ of power-bounded operators, $L_{cb}(X)$ of Cesaro-bounded operators, $\mathcal{E}(X)$*

of ergodic operators and $\mathcal{UE}(X)$ of uniformly ergodic operators are all Borel in $(L(X), \sigma(S_{op}))$.

Proof. Let $\{x_n\}_{n \in \mathbb{N}}$ be a dense sequence in the unit closed ball B_X of X . It is not difficult to show that

$$L_{pb}(X) = \bigcup_{k \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \{T \in L(X); \|T^m x_n\| \leq k\}.$$

It follows from the continuity of $T \mapsto T^n$ that $L_{pb}(X)$ is F_σ in $(L(X), \sigma(S_{op}))$. Similarly for the set $L_{cb}(X)$ using the S_{op} -continuity on bounded subsets of $L(X)$ of the maps $R \mapsto \frac{1}{n} \sum_{i=0}^{n-1} R^i$.

Let T be a bounded operator on X . By lemma 2.2, T is ergodic if and only if T is Cesaro-bounded and the sequence $\{(\frac{1}{n} \sum_{i=0}^{n-1} T^i)x_k\}_n$ is norm-Cauchy for all $k \in \mathbb{N}$; i.e.

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N \quad \left\| \frac{1}{n} \sum_{i=0}^{n-1} T^i x_k - \frac{1}{m} \sum_{i=0}^{m-1} T^i x_k \right\| < \varepsilon.$$

In other terms,

$$T \in \bigcap_{k \in \mathbb{N}} \bigcap_{p \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n, m \geq N} \left\{ R \in L(X) : \left\| \frac{1}{n} \sum_{i=0}^{n-1} R^i x_k - \frac{1}{m} \sum_{i=0}^{m-1} R^i x_k \right\| < \frac{1}{p} \right\}.$$

This proves the result for $\mathcal{E}(X)$ using the continuity of $T \mapsto T^n$.

The same arguments prove the result for $\mathcal{UE}(X)$ since T is uniformly ergodic if and only if T is Cesaro-bounded and the sequence $\{A_n\}_{n \in \mathbb{N}}$ of its Cesaro-means is Cauchy for the norm of $L(X)$. \square

3. SET OF SUPER-ERGODIC OPERATORS

First, an entropy-tree will be defined to help characterize the super-ergodicity of an operator. For an ultrafilter \mathcal{U} on \mathbb{N} , denote by $A_n^\mathcal{U}$ the n^{th} Cesaro-mean of the ultrapower $T_\mathcal{U}$:

$$A_n^\mathcal{U} = \frac{1}{n} \sum_{k=0}^{n-1} T_\mathcal{U}^k$$

By definition, an operator T is not super-ergodic if and only if there exist an ultrafilter \mathcal{U} and $\bar{x} \in X_\mathcal{U}$ such that $\{A_n^\mathcal{U} \bar{x}\}_n$ does not converge, i.e.,

$$\exists \bar{x} \in B_{X_\mathcal{U}}, \exists \varepsilon > 0, \exists J = \{j_p\}_p \in \mathbb{N}^{\uparrow \mathbb{N}} : \forall p \in \mathbb{N}, \left\| A_{j_p}^\mathcal{U} \bar{x} - A_{j_{p+1}}^\mathcal{U} \bar{x} \right\|_{X_\mathcal{U}} > \varepsilon$$

where $\mathbb{N}^{\uparrow \mathbb{N}}$ is the set of infinite and strictly increasing sequences of \mathbb{N} .

Let $(x_n)_{n \in \mathbb{N}} \in \bar{x}$ be chosen in the unit ball B_X . The condition

$$\forall p \in \mathbb{N}, \quad \left\| A_{j_p}^{\mathcal{U}} \bar{x} - A_{j_{p+1}}^{\mathcal{U}} \bar{x} \right\|_{X_{\mathcal{U}}} > \varepsilon$$

is then equivalent to

$$\forall p \in \mathbb{N}, \exists E_p \in \mathcal{U} : \quad \forall n \in E_p, \quad \left\| A_{j_p} x_n - A_{j_{p+1}} x_n \right\| > \varepsilon,$$

or again, by using $E_m = \bigcap_{p \leq m} E_p$,

$$\forall m \in \mathbb{N}, \exists E_m \in \mathcal{U} : \quad \forall n \in E_m, \quad \left\| A_{j_p} x_n - A_{j_{p+1}} x_n \right\| > \varepsilon \quad \forall p \leq m.$$

This implies in particular that

$$\forall m \in \mathbb{N}, \exists x_m \in B_X : \quad \left\| A_{j_p} x_m - A_{j_{p+1}} x_m \right\| > \varepsilon \quad \forall p \leq m.$$

Therefore, this proves that if T is not super-ergodic, then T satisfies the following condition, noted (\mathcal{NSE}) ;

$$\exists \varepsilon > 0, \exists J = \{j_p\}_{p \in \mathbb{N}} \in \mathbb{N}^{\uparrow \mathbb{N}}, \forall m \in \mathbb{N}, \exists x_m \in B_X :$$

$$\left\| A_{j_p} x_m - A_{j_{p+1}} x_m \right\| > \varepsilon \quad \forall p \leq m.$$

Lemma 3.1. *T is not super-ergodic if and only if T satisfies (\mathcal{NSE}) .*

Proof. One direction was proved above. Suppose now that T satisfies (\mathcal{NSE}) . Let \mathcal{U} be the ultrafilter on \mathbb{N} that contains all sets $E_n := \{n, n+1, \dots\}$. Put $\bar{x} = (x_m)_{m \in \mathbb{N}} + C_{\mathcal{U}}(X) \in X_{\mathcal{U}}$. The condition (\mathcal{NSE}) implies that

$$\forall m \in E_p, \quad \left\| A_{j_p} x_m - A_{j_{p+1}} x_m \right\| > \varepsilon.$$

So, $\mathcal{U} - \lim_m \left\| A_{j_p} x_m - A_{j_{p+1}} x_m \right\| > \frac{\varepsilon}{2}$, and thus $\left\| A_{j_p} \bar{x} - A_{j_{p+1}} \bar{x} \right\|_{\mathcal{U}} > \varepsilon$. Since this is true for any positive integer p , it follows that the sequence $\{A_n^{\mathcal{U}}(\bar{x})\}_{n \in \mathbb{N}}$ is not Cauchy and hence T is not super-ergodic. \square

With lemma 3.1, the super-ergodicity can be described in terms of trees using the following notations:

- $\mathbb{N}^{\uparrow < \mathbb{N}}$ denotes the set of finite and strictly increasing sequences in \mathbb{N} as well as the empty sequence.
- For $s \in \mathbb{N}^{\uparrow < \mathbb{N}}$, s_p denotes the p^{th} element of s and $|s|$ denotes the length of s .
- For $s, s' \in \mathbb{N}^{\uparrow < \mathbb{N}}$, $s \prec s'$ means that $|s| < |s'|$ and have the same first $|s|$ elements.

Definition 3.2. Let X be a Banach space and $T \in L(X)$ with A_n its n^{th} Cesaro-mean. For all $\varepsilon > 0$, $\mathcal{A}_\varepsilon(T, \varepsilon)$ is the tree on \mathbb{N} defined by the set of all elements $s \in \mathbb{N}^{\uparrow < \mathbb{N}}$ such that

$$|s| \leq 1 \text{ or } \exists x \in B_X \text{ such that } \forall 1 \leq p < |s|, \|A_{s_p}x - A_{s_{p+1}}x\| > \varepsilon.$$

It follows from Lemma 3.1 that T is not super-ergodic if and only if

$$\exists \varepsilon > 0 \text{ and } \exists J \in \mathbb{N}^{\uparrow \mathbb{N}} \text{ such that } \forall s \prec J, s \in \mathcal{A}(T, \varepsilon).$$

In other terms, T is not super-ergodic if and only if the tree $\mathcal{A}_\varepsilon(T, \varepsilon)$ is not well founded for certain $\varepsilon > 0$; i.e., with infinite branches.

Theorem 3.3. *Let X be a Banach space and $T \in L(X)$. The following assertions are equivalent:*

- (a) T is super-ergodic.
- (b) For all $\varepsilon > 0$, the tree $\mathcal{A}_\varepsilon(T, \varepsilon)$ is well founded.
- (c) $\eta_e(T) := \sup_{\varepsilon > 0} h(\mathcal{A}_\varepsilon(T, \varepsilon)) < \omega_1$, where h gives the height of a tree.

Proof. The equivalence between (a) and (b) was shown earlier. The equivalence between (b) and (c) is obvious because (b) is equivalent to

$$\forall n \in \mathbb{N}, \mathcal{A}_\varepsilon(T, \frac{1}{n}) \text{ is well founded.}$$

□

The index η_e defined in theorem 3.3 extends to all $T \in L(X)$ by $\eta_e(T) = \omega_1$, if T is not super-ergodic.

Theorem 3.4. *Let X be a separable Banach space, and $L(X)$ be the space of bounded operators equipped with the strong operator topology S_{op} . Let η_e be the index on $L(X)$ defined above. Then:*

- (a) T is super-ergodic if and only if $\eta_e(T) < \omega_1$.
- (b) The set $\mathcal{SE}(X)$ of super-ergodic operators is coanalytic.
- (c) η_e is a coanalytic rank on $\mathcal{SE}(X)$.
- (d) $\exists \alpha < \omega_1$ such that the set of uniformly ergodic operators $\mathcal{UE}(X) \subseteq \{T \in \mathcal{SE}(X); \eta_e(T) \leq \alpha\}$.
- (e) $\mathcal{SE}(X)$ is a Borel set if and only if $\eta_e(X) := \sup_{T \in \mathcal{SE}(X)} \eta_e(T) < \omega_1$.

Proof. The assertion (a) is part of the theorem 3.3. Let T be a bounded operator on X and A_n its n^{th} Cesaro-mean. We can write that

$$\eta_e(T) = \sup_{n \in \mathbb{N}} h\left(\mathcal{A}_\varepsilon\left(T, \frac{1}{n}\right)\right).$$

We need to construct a tree on \mathbb{N} that contains all trees $\mathcal{A}_\varepsilon(T, \frac{1}{n})$ while keeping the information on the index $\eta_e(T)$. Let $\mathcal{A}_e(T)$ be the tree

formed by the finite sequences $s = (s_0, s_1, s_2, \dots)$ such that s_0 covers \mathbb{N} and (s_1, s_2, \dots) covers the trees $\mathcal{A}_e(T, \frac{1}{s_0})$; i.e., the tree

$$\left\{ \sigma \in \mathbb{N}^{\uparrow < \mathbb{N}} : |\sigma| = 0 \text{ or } \sigma = (k, s) \in \mathbb{N} \times \mathbb{N}^{\uparrow < \mathbb{N}} \text{ with } s \in \mathcal{A}_e(T, \frac{1}{k}) \right\}.$$

Consider the map on the S_{op} -Borel set of Cesaro-bounded $L_{cb}(X)$,

$$\begin{aligned} \mathcal{A}_e : L_{cb}(X) &\longrightarrow \{\text{Trees on } \mathbb{N}\} \\ T &\longmapsto \mathcal{A}_e(T). \end{aligned}$$

Claim 3.5. Let $\sigma \in \mathbb{N}^{< \mathbb{N}}$. Put $\bar{\sigma} = \{T \in L_{cb}(X); \sigma \in \mathcal{A}_e(T)\}$. Then $\bar{\sigma}$ is a S_{op} -Borel subset of $L_{cb}(X)$.

Indeed, this is clear if the length of $|\sigma| \leq 2$ since in this case either $\bar{\sigma} = L_{cb}(X)$ or $\bar{\sigma} = \emptyset$. Let $\sigma = (k, s) \in \mathbb{N} \times \mathbb{N}^{< \mathbb{N}}$ with $|s| > 2$. If $s \notin \mathbb{N}^{\uparrow < \mathbb{N}}$ then $\bar{\sigma} = \emptyset$. If $s \in \mathbb{N}^{\uparrow < \mathbb{N}}$, we have

$$\begin{aligned} \bar{\sigma} &= \left\{ T \in L_{cb}(X) : s \in \mathcal{A}_e(T, \frac{1}{k}) \right\} \\ &= \left\{ T \in L_{cb}(X) : \exists x \in B_X, \forall 1 \leq p < |s|; \|A_{s_p}x - A_{s_{p+1}}x\| > \varepsilon \right\}. \end{aligned}$$

Since X is separable, let $\{x_n\}_{n \in \mathbb{N}}$ be dense in the unit ball of X . Then,

$$\begin{aligned} \bar{\sigma} &= \left\{ T \in L_{cb}(X) : \exists n \in \mathbb{N}, \forall 1 \leq p < |s|; \|A_{s_p}x_n - A_{s_{p+1}}x_n\| > \varepsilon \right\} \\ &= \bigcup_{n \in \mathbb{N}} \bigcap_{p=1}^{|s|-1} \left\{ T \in L_{cb}(X) : \|A_{s_p}x_n - A_{s_{p+1}}x_n\| > \varepsilon \right\}. \end{aligned}$$

It follows from the continuity of $T \mapsto T^n$ that $\bar{\sigma}$ is S_{op} -Borel.

Lemma 3.6 below and claim 3.5 imply that the set

$$C := \{T \in L_{cb}(X); \mathcal{A}_e(T) \text{ is well bounded} \}$$

is S_{op} -coanalytic in $L_{cb}(X)$ with a coanalytic rank $h \circ \mathcal{A}_e$; which maps T to the height of the tree $\mathcal{A}_e(T)$. On the other hand, by the definition of $\mathcal{A}_e(T)$ and theorem 3.3,

$$\begin{aligned} C &= \left\{ T \in L_{cb}(X); \mathcal{A}(T, \frac{1}{n}) \text{ is well founded } \forall n \in \mathbb{N}^* \right\} \\ &= \left\{ T \in L_{cb}(X); \mathcal{A}(T, \varepsilon) \text{ is well founded } \forall \varepsilon > 0 \right\} \\ &= \mathcal{SE}(X) \cap L_{cb}(X) \\ &= \mathcal{SE}(X). \end{aligned}$$

Therefore, the set $\mathcal{SE}(X)$ is S_{op} -coanalytic in $L_{cb}(X)$ and thus in $L(X)$ because $L_{cb}(X)$ is a S_{op} -Borel subset of $L(X)$, which proves (b). The index η_e is then a coanalytic rank on $\mathcal{SE}(X)$ since $\eta_e = h \circ \mathcal{A}_e$, thus

proving (c). The assertions (d) and (e) of the theorem follow from the lemma 3.7 below on coanalytic ranks.

□

Below are adaptations, in particular to the topology on the set of all trees on \mathbb{N} , of classical results of descriptive set theory used in the proof above (see [6] and [12]).

Lemma 3.6. *Let P be a Polish space and ψ be a map from P into the set of all trees on \mathbb{N} . If for all $s \in \mathbb{N}^{<\mathbb{N}}$, the set $\bar{s} = \{x \in P : s \in \psi(x)\}$ is Borel, then the set $C = \{x \in P; \psi(x) \text{ well founded}\}$ is coanalytic in P with $h \circ \psi$ as coanalytic rank.*

Lemma 3.7. *Let δ be a coanalytic rank on a coanalytic subset C of a Polish space P . Then:*

- (a) $\forall \alpha < \omega_1$, $B_\alpha := \{x \in C; \delta(x) \leq \alpha\}$ is a Borel set.
- (b) If $A \subseteq C$ is analytic, then $\exists \alpha < \omega_1$ such that $A \subseteq B_\alpha$.
- (c) C is Borel if and only if δ is bounded on C by a countable ordinal.

The topological hierarchy of these families of operators makes it possible to put strong limits on possible relationships among using the index $\eta_e(X)$ of a space X or the rank $\eta_e(T)$ of an operator T . This generates families of Banach spaces with a desired relationship instead of just individual examples. In particular, if $\eta_e(X) = \omega_1$, then the set of super-ergodic operators strictly separates the sets of ergodic and uniformly ergodic operators. Moreover, not only is an operator T super-ergodic when its rank $\eta_e(T)$ is countable, we also have the interesting dichotomy below. It is an application of the facts that η_e is a coanalytic rank and the set of ergodic operator is Borel.

Corollary 3.8. *For every separable Banach space X exactly one of the following holds:*

- either there exists an ergodic but not super-ergodic operator on X ,
- either there exists a countable ordinal α such that for any ergodic operator T on X , $\eta_e(T) \leq \alpha$.

In particular, for any Banach space that contains a complemented $\ell^1(\mathbb{N})$, there exists ergodic operators with arbitrary large ordinals. Indeed, the example 3.3 in [11] shows that the left-shift operator S on $\ell^1(\mathbb{N})$ is ergodic since $\lim_{n \rightarrow \infty} \|S^n x\| = 0$, but it is not super-ergodic since the $\{A_n^U(\bar{e})\}_{n \in \mathbb{N}}$ is not Cauchy; where A_n^U is the n^{th} Cesaro-mean of S_U and \bar{e} is the classical canonical basis $(e_k)_{k \in \mathbb{N}}$.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, URSINUS COLLEGE, COLLEGEVILLE, PA 19426, USA

E-mail address: myahdi@ursinus.edu